Effects of Boundary Conditions on the Stability of Cylinders Subject to Lateral and Axial Pressures

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This paper presents an analysis of critical combinations of uniform external lateral pressure and central axial compressive force for circular cylindrical shells with arbitrarily prescribed boundary conditions. Numerical results are presented for the special case of hydrostatic pressure loading for eight different sets of boundary conditions. The numerical results show that prevention of edge rotation of a generator has little effect on the critical pressure unless the cylinder is very short. On the other hand, it is shown that the presence of axial restraint at the boundaries significantly increases critical pressures for intermediate-length cylinders.

Nomenclature

A_i, B_i = displacement constants [see Eqs. (3)]
a = mean radius of undeformed cylinder
D = $Eh^3/12(1-\nu^2)$
E = Young's modulus
F = axial compressive force
h = thickness of cylinder
L = length of cylinder
$M_{x1}, M_{x\varphi 1}, N_{x1}, N_{x\varphi 1}$ = incremental stress resultants due to buckling (see Fig. 2)
n = number of circumferential waves in buckle pattern
p = lateral pressure; positive inward
p_a = $F/2\pi ah$
p_{cr} = critical lateral pressure
r_i = roots of Eqs. (6)
T_{x1} = effective shear force at end of cylinder
$U_n(x), V_n(x), W_n(x)$ = displacement functions [see Eqs. (2)]
u_i, v_i = displacement constants [see Eqs. (4) and (5)]
u, v, w = nondimensional incremental axial, circumferential, and radial displacement components of a point in the cylinder's middle surface; corresponding distances represented by au, av, and aw, respectively; w positive outward
x, φ = nondimensional axial and circumferential coordinates of a point P on the undeformed cylinder's middle surface; corresponding axial distance represented by ax (see Fig. 2)
α = L/a
β^2 = $12(1-\nu^2)\gamma^2$
γ = a/h
λ = pa/Eh, nondimensional lateral pressure parameter
λ_a = p_a/E, nondimensional axial pressure parameter
ν = Poisson's ratio
ω = $2\lambda_a/\lambda(\lambda \neq 0)$, pressure ratio coefficient ($\omega = 1$ for hydrostatic pressure loading, and $\omega = 0$ for lateral pressure loading)
$\nabla^2(\ )$ = $(\ )_{,xx} + (\ )_{,\varphi\varphi}$
x, φ = as subscripts, after comma, refer to partial differentiation of principal symbol with respect to x or φ.

Introduction

It is generally acknowledged that von Mises[1] has given a satisfactory solution for the critical hydrostatic pressure of a simply supported circular cylindrical shell. Such a settled state of affairs does not exist for the corresponding problem of the clamped cylinder. Analyses of the clamped cylinder have been made by Nash,[2] Galletly and Bart,[3] and Palyi.[4] A one-degree-of-freedom Rayleigh-Ritz approach was used in Refs. 2 and 4, whereas in Ref. 3 a one-degree-of-freedom Galerkin approach was used in conjunction with the modified Donnell equation. The results of Refs. 2 and 3 give critical pressures that are considerably higher than those for simply supported cylinders (see Fig. 1). However, it seems unlikely that the effect of edge rotational restraint would be significant for relatively long thin cylinders. Palyi, in an attempt to improve Nash's results, used a radial deflection function w that has a sharp curvature at the ends of the cylinder and which, in the middle, corresponds in shape to that for a simply supported cylinder. His results fall between those of Nash and those of von Mises and are considerably closer to the latter unless the shell is quite short (see Fig. 1). The results obtained by Singer for a cylinder with axial elastic springs but without rotational restraint[5] indicate that axial restraint at the edges will significantly increase the critical pressure. Now, the analyses of Refs. 2 and 3 are applicable to axially restrained clamped cylinders, whereas the simply supported cylinder analyzed in Ref. 1 does not have axial restraint. Hence, it appears that the presence of axial restraint in the clamped cylinder analyzed in Refs. 2 and 3 might account, to a large extent, for the higher critical pressures. In view of these investigations, it seems desirable to perform an analysis in which all types of boundary conditions can be considered. This is the aim of the present paper. Such an analysis is easily effected by assuming, as was done in Refs. 1–4, that the cylinder configuration at impending buckling can be approximated by a homogeneous membrane state of stress. Because of this simplification, the governing partial differential equations can be reduced to ordinary linear differential equations with con-

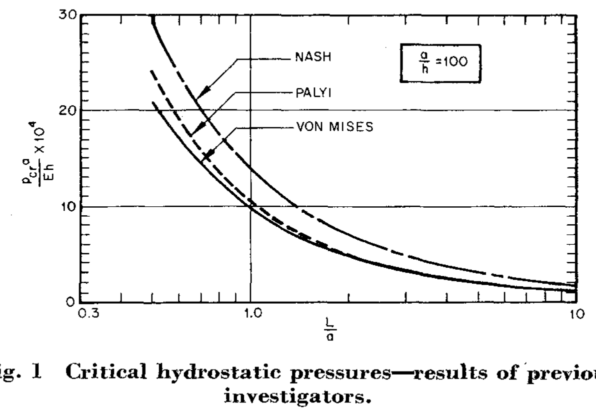

Fig. 1 Critical hydrostatic pressures—results of previous investigators.

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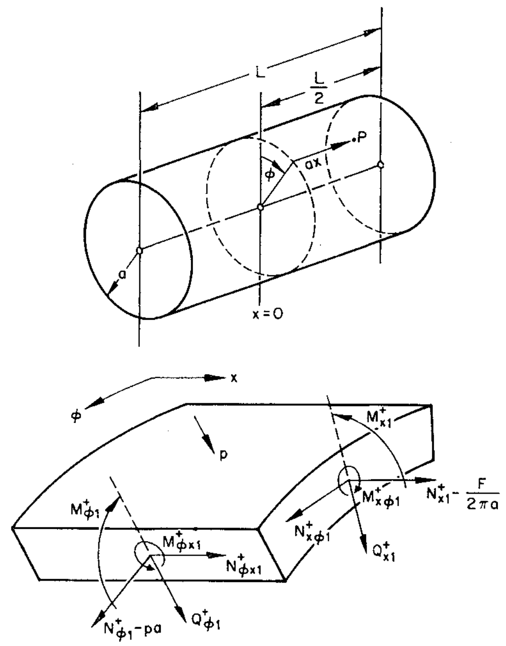

Fig. 2 Coordinate system and shell element.

stant coefficients, and solutions can be obtained in a straightforward manner.

The present analysis is formulated for independent axial and lateral loadings. For convenience, the terms "simply supported" and "clamped" are here used to designate cylinders for which $w_{,xx}$ and $w_{,x}$, respectively, vanish at both ends of the cylinder. This designation, as used here, does not imply the vanishing of any other prescribed quantity such as u, v, N_{x1}, or T_{x1}. Four sets of boundary conditions consisting of combinations of incremental in-plane displacement components (u, v) and incremental stress resultants (N_{x1}, T_{x1}) are considered for both simply supported and clamped cylinders.

Theoretical Analysis

The coupled Donnell equations for the three incremental displacement components are[6] (see Fig. 2)

$$\left.\begin{aligned} &\frac{1}{\beta^2}\nabla^4 w + \frac{1}{1-\nu^2}\times \\ &\qquad (v_{,\varphi} + w + \nu u_{,x}) + \lambda w_{,\varphi\varphi} + \lambda_a w_{,xx} = 0 \\ &u_{,xx} + \frac{(1-\nu)}{2}u_{,\varphi\varphi} + \frac{(1+\nu)}{2}v_{,x\varphi} + \nu w_{,x} = 0 \\ &v_{,\varphi\varphi} + \frac{(1-\nu)}{2}v_{,xx} + \frac{(1+\nu)}{2}u_{,x\varphi} + w_{,\varphi} = 0 \end{aligned}\right\} \quad (1)$$

By expanding the incremental displacement components into a Fourier series in the circumferential coordinate φ, we satisfy the requirement of periodicity in φ and separate the space variables x, φ in Eqs. (1).

When we let

$$\left.\begin{aligned} w(x, \varphi) &= \sum_{n=0}^{\infty} W_n(x)\cos(n\varphi) \\ u(x, \varphi) &= \sum_{n=0}^{\infty} U_n(x)\cos(n\varphi) \\ v(x, \varphi) &= \sum_{n=1}^{\infty} V_n(x)\sin(n\varphi) \end{aligned}\right\} \quad (2)$$

each expression of Eqs. (1) generates an infinite set of linear homogeneous ordinary differential equations for the Fourier functions $U_n(x)$, $V_n(x)$, and $W_n(x)$. Fortunately, the various harmonics uncouple for each of Eqs. (1), so that we need consider only the general harmonic n and thereby obtain a particular solution of Eqs. (1):

$$\left.\begin{aligned} w_n(x, \varphi) &= \cos(n\varphi)\sum_{i=1}^{4}\{A_i\cosh[(r_i)^{1/2}x] + \\ &\qquad B_i\sinh[(r_i)^{1/2}x]\} \\ u_n(x, \varphi) &= \cos(n\varphi)\sum_{i=1}^{4}\{u_iA_i\sinh[(r_i)^{1/2}x] + \\ &\qquad u_iB_i\cosh[(r_i)^{1/2}x]\} \\ v_n(x, \varphi) &= \sin(n\varphi)\sum_{i=1}^{4}\{v_iA_i\cosh[(r_i)^{1/2}x] + \\ &\qquad v_iB_i\sinh[(r_i)^{1/2}x]\} \end{aligned}\right\} \quad (3)$$

where

$$u_i = \frac{-(r_i)^{1/2}(n^2 + \nu r_i)}{(r_i - n^2)^2} \quad (4)$$

$$v_i = \frac{n[(2+\nu)r_i - n^2]}{(r_i - n^2)^2} \qquad i = 1, \ldots, 4 \quad (5)$$

and $r_i (i = 1, \ldots, 4)$ are the roots of the equations

$$\left.\begin{aligned} &(r - n^2)^4 + [-n^2\beta^2\lambda + r\beta^2(\omega/2)\lambda]\times \\ &\qquad (r - n^2)^2 + \beta^2 r^2 = 0 \\ &\text{whenever} \\ &\qquad p \neq 0 \\ &\text{or} \\ &\qquad (r - n^2)^4 + r\beta^2\lambda_a(r - n^2)^2 + \beta^2 r^2 = 0 \\ &\text{if} \\ &\qquad p = 0 \end{aligned}\right\} \quad (6)$$

The specification for four linear, homogeneous boundary conditions at each end of the cylinder and utilization of Eqs. (3) result in a system of eight linear, homogeneous algebraic equations for the eight arbitrary constants A_i and $B_i (i = 1, \ldots, 4)$. The possible buckled shell configurations are given by the nontrivial solutions to this system. For fixed values of the geometric parameters α and β, the pressure ratio coefficient ω, and the number of circumferential waves n, the coefficients in this system depend on the load parameter λ (or λ_a). However, nontrivial solutions can exist only for certain discrete values of λ, namely, those values of λ which cause the determinant of the coefficients $\Delta(\lambda)$ to vanish. Thus, for a fixed n, we seek the smallest value of λ for which $\Delta = 0$. Then by varying n, we obtain a set of these smallest values of λ, one for each n, and the solution to the buckling problem is given by the minimum λ in this set.

Now we shall show that, although the elements $D_{kl}(k, l = 1, \ldots, 8)$ of Δ are, in general, complex quantities, the value of Δ is always either real or purely imaginary. This result will be of importance later when we use an iterative technique to obtain the load parameter λ. We first use the definition of a determinant to prove that $\overline{|D_{kl}|} = |\overline{D_{kl}}|$ and then use the principle of reflection[7] to show that complex conjugate roots of the quartic equation [Eqs. (6)] produce complex conjugate columns in Δ. Then, for the case where all the roots of Eqs. (6) are complex, it is easy to show that $\Delta = \bar{\Delta}$, which implies that Δ is real. If two of the roots are complex, we find that $\Delta = -\bar{\Delta}$, and thus Δ is purely imaginary.

We note that the stability determinant Δ will always vanish whenever two of the roots of the quartic equation [Eqs. (6)] are equal. For a given cylinder, there exists a certain value of the load parameter (λ^* or λ_a^*) for which the quartic equation possesses two equal roots. Moreover, this

Table 1 Critical hydrostatic pressures $(p_{cr}a/Eh) \times 10^3$, $a/h = 100$

	Present Analysis								Galletly & Bart	Nash
	$N_{x1} = 0$, $w = 0$				$u = 0$, $w = 0$					
①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪
L/a	S_1	C_1	S_2	C_2	S_3	C_3	S_4	C_4		
	$w_{,xx} = 0$, $v = 0$	$w_{,x} = 0$, $v = 0$	$w_{,xx} = 0$, $T_{x1} = 0$	$w_{,x} = 0$, $T_{x1} = 0$	$w_{,xx} = 0$, $T_{x1} = 0$	$w_{,x} = 0$, $T_{x1} = 0$	$w_{,xx} = 0$, $v = 0$	$w_{,x} = 0$, $v = 0$	$w_{,x} = 0$, $v = 0$	$w_{,x} = 0$, $v = 0$
0.5	2.106 (11)[a]	2.652 (12)	1.930 (10)	2.632 (11)	2.078 (10)	2.798 (12)	2.366 (11)	2.851 (12)	2.912 (12)	2.930 (12)
0.6	1.715 (10)	2.068 (11)	1.599 (9)	2.065 (11)	1.771 (10)	2.239 (11)	1.976 (11)	2.282 (11)	2.349 (11)	2.366 (11)
0.75	1.344 (9)	1.557 (10)	1.268 (9)	1.557 (10)	1.455 (9)	1.743 (10)	1.599 (10)	1.774 (10)	1.846 (10)	1.862 (10)
1.0	0.9838 (8)	1.104 (8)	0.9416 (8)	1.103 (8)	1.146 (8)	1.289 (9)	1.224 (9)	1.306 (9)	1.372 (9)	1.388 (9)
1.5	0.6417 (7)	0.6891 (7)	0.6290 (7)	0.6889 (7)	0.8094 (7)	0.8636 (8)	0.8446 (8)	0.8688 (8)	0.9075 (8)	0.9213 (8)
2.0	0.4744 (6)	0.5023 (6)	0.4657 (6)	0.5020 (6)	0.6250 (7)	0.6520 (7)	0.6441 (7)	0.6552 (7)	0.6833 (7)	0.6971 (7)
3.0	0.3132 (5)	0.3255 (5)	0.3097 (5)	0.3251 (5)	0.4342 (6)	0.4434 (6)	0.4412 (6)	0.4444 (6)	0.4580 (6)	0.4708 (6)
4.0	0.2395 (4)	0.2479 (4)	0.2350 (4)	0.2478 (4)	0.3273 (5)	0.3340 (5)	0.3333 (5)	0.3351 (5)	0.3466 (5)	0.3608 (5)
5.0	0.1878 (4)	0.1920 (4)	0.1867 (4)	0.1917 (4)	0.2764 (5)	0.2783 (5)	0.2779 (5)	0.2785 (5)	0.2830 (5)	0.2946 (5)
6.0	0.1679 (4)	0.1703 (4)	0.1676 (4)	0.1700 (4)	0.2246 (4)	0.2281 (4)	0.2281 (4)	0.2288 (4)	0.2364 (4)	0.2520 (4)
7.0	0.1378 (3)	0.1407 (3)	0.1361 (3)	0.1407 (3)	0.1923 (4)	0.1938 (4)	0.1936 (4)	0.1940 (4)	0.1978 (4)	0.2109 (4)
8.0	0.1158 (3)	0.1177 (3)	0.1151 (3)	0.1176 (3)	0.1751 (4)	0.1757 (4)	0.1756 (4)	0.1758 (4)	0.1779 (4)	0.1897 (4)
9.0	0.1038 (3)	0.1051 (3)	0.1035 (3)	0.1050 (3)	0.1653 (4)	0.1656 (4)	0.1655 (4)	0.1656 (4)	0.1669 (4)	0.1780 (4)
10.0	0.09678 (3)	0.09776 (3)	0.09662 (3)	0.09762 (3)	0.1423 (3)	0.1440 (3)	0.1442 (3)	0.1445 (3)	0.1492 (3)	0.1677 (3)

[a] Numbers in parentheses indicate the number of circumferential waves.

value of the load parameter is the same for all different sets of boundary conditions. For the case of lateral pressure loading only, λ^* is the critical pressure for an infinite cylinder. For the case of axial pressure loading only, $\lambda_a{}^*$ gives the classical critical stress $(0.6E\ h/a)$ for an infinite cylinder.

Four sets of boundary conditions consisting of combinations of incremental in-plane displacement components (u, v) and incremental stress resultants (N_{x1}, T_{x1}) are considered for both simply supported $(w = w_{,xx} = 0)$ and clamped $(w = w_{,x} = 0)$ cylinders. The eight sets of boundary conditions are designated by $S_1, \ldots, S_4, C_1, \ldots, C_4$, and are defined below.

The boundary conditions for simply supported cylinders are

$$\left.\begin{aligned} S_1&: w = w_{,xx} = N_{x1} = v = 0 \\ S_2&: w = w_{,xx} = N_{x1} = T_{x1} = 0 \\ S_3&: w = w_{,xx} = u = T_{x1} = 0 \\ S_4&: w = w_{,xx} = u = v = 0 \end{aligned}\right\} \tag{7}$$

The boundary conditions for clamped cylinders are

$$\left.\begin{aligned} C_1&: w = w_{,x} = N_{x1} = v = 0 \\ C_2&: w = w_{,x} = N_{x1} = T_{x1} = 0 \\ C_3&: w = w_{,x} = u = T_{x1} = 0 \\ C_4&: w = w_{,x} = u = v = 0 \end{aligned}\right\} \tag{8}$$

where

$$\left.\begin{aligned} N_{x1} &= \frac{Eh}{1-\nu^2}\,[u_{,x} + \nu(v_{,\varphi} + w)] \\ N_{x\varphi 1} &= \frac{Eh}{2(1+\nu)}\,[u_{,\varphi} + v_{,x}] \\ M_{x1} &= (D/a)[w_{,xx} + \nu w_{,\varphi\varphi}] \\ M_{x\varphi 1} &= (1-\nu)(D/a)w_{,x\varphi} \\ T_{x1} &= N_{x\varphi 1} \end{aligned}\right\} \tag{9}$$

Numerical Analysis

In this section we present numerical results for the special case of hydrostatic pressure loading $(\omega = 1)$. By considering only cylinders that have the same set of boundary conditions at each end of the shell, we can equate to zero the arbitrary constants $B_i (i = 1, \ldots, 4)$ in Eqs. (3).

The numerical results were obtained from a digital computer by use of an iterative technique. For fixed values of the geometric parameters α, β and the number of circumferential waves n, we assume different values $\bar{\lambda}$ for the load parameter λ and compute the corresponding values of $\Delta(\bar{\lambda})$ until we find two successive values $\bar{\lambda}_1$ and $\bar{\lambda}_2$ for which the signs of $\Delta(\bar{\lambda}_1)$ and $\Delta(\bar{\lambda}_2)$ are opposite. Thus, the solution λ is now isolated between $\bar{\lambda}_1$ and $\bar{\lambda}_2$. Next, by using the method of regula falsi in the interval $[\bar{\lambda}_1, \bar{\lambda}_2]$, we may compute the solution λ to any degree of accuracy. (Four significant figures were used here.) Thus, we obtain a solution accurate to four significant figures. It should be noted that, in the vicinity of the solution, the roots of the quartic equation [Eqs. (6)] consist of two real roots with opposite signs and two complex conjugate roots.

Numerical results for the eight sets of boundary conditions, Eqs. (7) and (8), are given in Table 1 for different values of L/a with a/h fixed at 100 and Poisson's ratio equal to 0.3. The set of boundary conditions represented by $S_1(w = w_{,xx} = N_{x1} = v = 0)$ is the same as that used in von Mises' solution for a simply supported cylinder. The results of the present paper, for the set S_1, agree with von Mises' results to four significant figures (the degree of accuracy used in the present analysis). Table 1 also gives the results obtained by Nash[2] and by Galletly and Bart[3] for clamped cylinders. The combination of boundary conditions used in Refs. 2 and 3 is here represented by the set $C_4(w = w_{,x} = u = v = 0)$. A comparison of columns 9 and 10 of Table 1 shows that the critical pressures for the set C_4 are always less than, but close to, the critical pressures given in Ref. 3.

The two curves in Fig. 3 give the critical pressures for a clamped cylinder and a simply supported cylinder represented by the sets of boundary conditions $C_4(w = w_{,x} = u = v = 0)$ and $S_4(w = w_{,xx} = u = v = 0)$, respectively. From these curves, we see that clamping does not significantly increase the critical hydrostatic pressures unless the cylinders are quite short. Moreover, this same result was always obtained whenever critical pressures for clamped cylinders were compared with those for simply supported cylinders

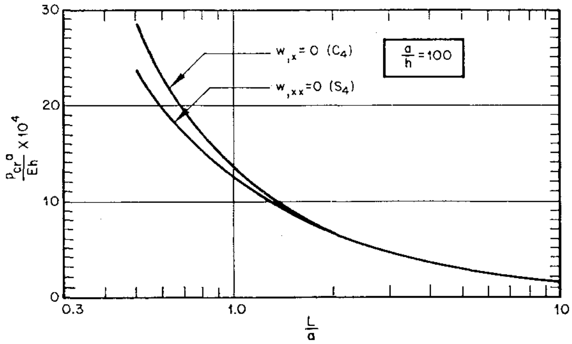

Fig. 3 Effect of rotational constraint on critical hydrostatic pressures.

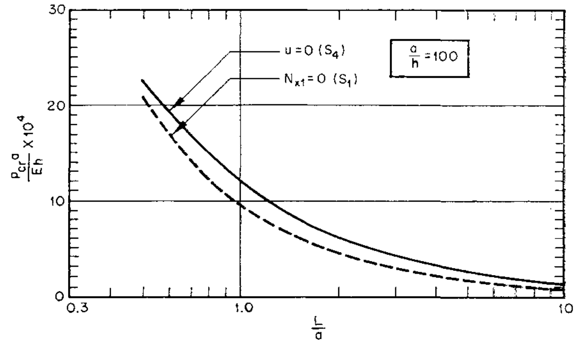

Fig. 4 Effect of axial constraint on critical hydrostatic pressures.

that have the same prescribed in-plane displacement components and stress resultants; hence, Fig. 3 also (qualitatively) depicts the effect of clamping for all such shells.

Now we compare the critical pressures of two cylinders for which the incremental axial displacement component $u(\alpha/2, \varphi)$ vanishes for the first cylinder and the incremental axial force $N_{x1}(\alpha/2, \varphi)$ vanishes for the second; the other boundary conditions are the same for both cylinders. Typical solutions are given in Fig. 4, in which the upper curve represents the solution for a cylinder with $u(\alpha/2, \varphi) = 0$, and the lower curve is for a cylinder with $N_{x1}(\alpha/2, \varphi) = 0$. From these curves, we see that the presence of axial restraint at the boundaries significantly increases critical pressures for intermediate-length cylinders.

From these observations, we can now explain why, even for long thin cylinders, the critical pressures given by Nash[2] and by Galletly and Bart[3] are considerably higher than those given by von Mises[1]: the clamped cylinder analyzed in Refs. 2 and 3 was axially restrained at its edges, whereas the simply supported cylinder analyzed in Ref. 1 was free to move axially.

We conclude by noting that Forsberg[8] recently has investigated the influence of boundary conditions on the free vibrations of cylinders. The approach used in Ref. 8 is identical to the one used here, and, as might be expected, the results of the stability and vibration studies lead to the same conclusions regarding the influence of various boundary conditions.

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